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Transfer matrix formulation of AC magnetoconductance

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Abstract. Within the linear response theory we derive an expression for the alternating current (AC) magnetoconductance (MC) of a quasi-one-dimensional mesoscopic system in terms of its pertinent transfer matrices. It is then employed in numerical simulations of the AC MC for some specific systems. At low frequencies and for weak magnetic fields, the AC MC of a clean system is expressible as a sum of independent contributions from the different one-dimensional subbands (which is trivially valid for DC conductance). For disordered systems, it is conjectured that neither the conductance nor its logarithm is self-averaging. Moreover, the fundamental phenomenon of positive MC no longer holds at finite frequencies.

1. Introduction

Recent experimental work [1] on the frequency dependence of the Aharonov–Bohm effect in Ag rings motivated the study of the dynamic response in mesoscopic systems. In recent studies [2], a self-consistent theory which relates the AC conductance to the scattering matrix in multi-port conductors has been developed. It has already been applied [3] in the investigation of dynamic magnetoconductance (MC) fluctuations in disordered metallic systems.

In this work, a somewhat related formalism aimed at the study of the dynamic response of mesoscopic systems subject to a constant magnetic field is presented, based on transfer matrix theory. In the absence of a magnetic field, the main steps of the present approach have been briefly explained in two earlier studies [4]. Therein, the AC conductance $G_L(E,\omega)$ at the Fermi energy E and frequency ω has been investigated for a two-probe non-interacting quasi-one-dimensional system of length L. The central object of the present work is an extension of the formalism to include the effect of the magnetic field, and the evaluation of the AC MC in several simple (but still interesting) cases. In particular, we are interested in the combined effects of disorder, magnetic field and finite-frequency electric field.

Admittedly, the formalism suggested below is less general than the complete theory detailed in [2], at least in two senses. First, it is useful mainly for two-probe systems with quasi-one-dimensional geometry. Second, it is primarily effective for non-interacting systems (the importance of the electron–electron interaction in the study of response functions has been stressed repeatedly in [2, 3]). Yet, one may, at least in principle, account for interaction effects within some screening approximation, since an exact treatment of the many-body problem is out of reach anyhow.

The main advantage of the present approach is its relative simplicity. The use of transfer matrix formulation proves to be very useful both for understanding the underlying physics as well as in actual numerical computation, especially for disordered systems. While much progress has been recorded in applying transfer matrix techniques to the study of DC

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conductance, it is still not in frequent use in the investigation of AC conductance. This is not surprising, since, in that case, it is not sufficient to know the total transfer matrix which transforms the wavefunction from one side of the sample to the other. Inspection of the Kubo formula shows that, in order to evaluate the AC conductance, the wavefunctions and the currents must be evaluated inside the probed system. Yet, this information is available anyhow when the total transfer matrix is computed, since one has to know the intermediate transfer matrices which transform the wavefunction from one point to another within the system.

It should be reiterated here that, conceptually, the transfer matrix theory is by no means a mere technical concept. Transfer matrix algorithms are reputed to be a very important tool in the investigation of the DC conductance of mesoscopic systems. Many physical concepts such as weak localization, positive MC, universal conductance fluctuations, universal corrections to localization length, etc, are elegantly analysed in terms of random transfer matrix theory [5, 6]. One of the questions which is addressed here concerns the effect of the magnetic field on the AC conductance of a weakly disordered system. As far as DC conductance is concerned, the theory of weak localization tells us that, in the absence of spin–orbit interaction, the magnetic field leads to an enhancement of the conductance [7]. This remarkable observation is referred to as a positive MC. To the best of our knowledge, the question of whether this phenomenon remains valid also for AC conductance has not yet been addressed. Our results explained below show that the positive MC deteriorates as the frequency increases, until it no longer holds.

This paper is organized as follows. In section 2, the formalism is developed for the tight-binding model and the expression for the AC MC is summarized in equations (2.19) and (2.20). In section 3 we apply this formulation for various quasi-one-dimensional systems. First we display the AC MC of a clean system in a continuous (strip) geometry (where the role of edge states is stressed) as well as for a two-chain tight-binding model. Then we concentrate on disordered systems and discuss two questions, namely the behaviour of the AC conductance as a function of the system's length and the question of whether the concept of a positive MC makes sense also for finite frequencies. A short summary is given in section 4.

2. The AC MC in terms of transfer matrices

In this section the derivation of an expression for the AC MC for quasi-one-dimensional lattice models governed by tight-binding Hamiltonians is carried out. (For simplicity we shall work in two-dimensional space.) Usually, transfer matrices in such systems are not given in the plane-wave representation, and hence they do not share the current conservation property. Hence, at the beginning of this section some notation and transfer matrices in the plane-wave representation are introduced for self-consistency. After this task is completed, the expression for the AC MC is presented. At the end we shall also comment on the formulation which is pertinent to continuous geometry (more precisely, to a system having the shape of a two-dimensional strip with parabolic confinement in the transverse direction).

Consider a square lattice (with lattice constant $a = 1$) in the form of an infinite strip whose sites are labelled by (j, m) with $-\infty < j < \infty$ and $m = -(M-1)/2, -(M-1)/2+1, \ldots, (M+1)/2$. On this lattice, an electron is hopping in the presence of a perpendicular magnetic field B. For $1 \leqslant j \leqslant L$ there is an interaction region (referred to as the sample) in which the electron is subject on each site to an elastic scattering. The elastic on-site potential is denoted by ε_{jm}, the distribution of which determines the nature of the quantum

eigenstates. The dynamics of the electron is governed by the tight-binding Hamiltonian [8]

$$H = \sum_j \left(\sum_m [\varepsilon_{jm}|jm\rangle\langle jm| + (|jm\rangle\langle jm+1| + V\exp(im\phi)|jm\rangle\langle j+1m| + h.c)] \right) \quad (2.1)$$

where $\{|jm\rangle\}$ is a complete set of orthonormal and localized states associated with the lattice sites (j, m). The hopping matrix element between nearest neighbours in the y direction is taken to be the unit of energy, while that between nearest neighbours in the x direction is V. As usual, if we need to investigate impure systems, the disorder is implemented through the site potentials ε_{jm} which are considered to be uncorrelated random variables. We use a hard wall boundary on the transverse (m) direction so that in the Landau gauge the vector potential on the horizontal link is $m\phi$, where $\phi = eB/\hbar$ denotes the number of magnetic flux quanta in one unit cell times 2π. Starting with the Schrödinger equation $H\Psi = E\Psi$, the wavefunction is expanded in terms of lattice site states:

$$\Psi = \sum_{jm} c_{jm}|jm\rangle \quad (2.2)$$

where c_{jm} are complex coefficients, for which the following set of equations is obtained:

$$\varepsilon_{jm}c_{jm} + c_{jm+1} + c_{jm-1} + V(\exp(im\phi)c_{j+1m} + \exp(-im\phi)c_{j-1m}) = Ec_{jm}. \quad (2.3)$$

Let us first recall the main results in the simple (ordered) case with ε_{jm}. To simplify the notation, denote by c_j the M-dimensional column vector of components c_{jm}, by Φ the diagonal matrix with diagonal elements $\exp(im\phi)$ and by X the symmetric matrix with 1 on both sides of the diagonal and zero otherwise. Then one can rewrite equation (2.3) in the form

$$V(\Phi c_{j+1} + \Phi^\dagger c_{j-1}) + Xc_j = Ec_j. \quad (2.4)$$

With the separation $c_j = \exp(ikj)Y$ with $-\pi < k < \pi$, one defines Y as an M-dimensional vector whose components $Y(m)$ (independent of j) satisfy the Harper equation

$$[2V\cos(k + A) + X - E]Y = 0 \quad (2.5)$$

where $\cos(k + A)$ is a diagonal matrix of dimension M with diagonal elements $\cos(k + m\phi)$.

When equation (2.5) is considered as an eigenvalue equation, the solutions $Y_n(m)$ and the dispersion curves $E_n(k)$ are sought (here n is the band index). In a scattering problem such as that discussed here, the problem is reversed in the sense that the energy E is given inside a band, and one looks for the corresponding momenta. These are the solutions of the equations

$$E_n(k_n) = E. \quad (2.6)$$

The solutions of equation (2.5) at the given scattering energy E and an allowed momentum k_n are denoted by $Y_n(k_n; m)$. When k_n is real, the wavefunction

$$|\Psi_n\rangle = \frac{1}{\sqrt{N_n}} \sum_{jm} c_{jm}^n |jm\rangle = \frac{1}{\sqrt{N_n}} \sum_{jm} \exp(isk_n j)Y_n(k_n; m)|jm\rangle \quad (2.7)$$

is a current-carrying (propagating) state (where N_n is a normalization constant to be determined below and $s = \pm 1$ depending on the initial direction of the incoming wave). For any solution in channel n with $k_n > 0$ there is another with $k_n < 0$ (due to the reflection symmetry of our system about the horizontal axis). The normalization constant N_n is fixed by the condition that the surface integral of the current density of the propagating states is

unity. Note that the velocity operator in the tight-binding model is $\sin(k + A)$, and so we require that

$$\frac{1}{N_n} \sum_{m=-M/2}^{M/2} c_{jm}^{n*} \sin(k_n + m\phi) c_{jm}^{n} = 1. \tag{2.8}$$

This condition together with equation (2.5) yields

$$N_n = \frac{1}{2V} \left(\frac{\mathrm{d}E_n}{\mathrm{d}k} \right)_{k=k_n}. \tag{2.9}$$

With these definitions, it is not difficult to derive an expression for the off-diagonal matrix element of the current [9]. Furthermore, only with the above definition of the current can one show from equation (2.3) that the following orthogonality relation holds:

$$\sum_{m=-M/2}^{M/2} c_{jm}^{n*} \sin\left(\frac{k_n + k_{n'}}{2} + m\phi \right) c_{jm}^{n'} = \delta_{nn'} I_n \tag{2.10}$$

where I_n is hence defined.

The standard application of transfer matrix employed in the present context is to obtain the $2M$-component vector (c_{j+1}, c_j) from the $2M$-component vector (c_j, c_{j-1}). Here, however, we need a different representation. Let us expand the wavefunction in a plane-wave basis:

$$c_{jm} = a_{jm} \exp(\mathrm{i}q_m j) + b_{jm} \exp(-\mathrm{i}q_m j) \tag{2.11}$$

where $q_m = k_m + m\phi$ and k_m is defined in equation (2.6). The $2M \times 2M$ transfer matrix in the plane-wave representation, denoted here by $\tau^{(j)}$, transforms the $2M$-component vector (a_j, b_j) to the $2M$-component vector (a_{j+1}, b_{j+1}). Straightforward (but somewhat tedious) algebra results its explicit form

$$\tau^{(j)} = I + \frac{\mathrm{i}}{2} \begin{bmatrix} \epsilon(j) & 0 \\ 0 & \epsilon(j) \end{bmatrix} \begin{bmatrix} \alpha & \beta(j)^* \\ -\beta(j) & -\alpha \end{bmatrix} \tag{2.12}$$

where I is the $2M \times 2M$ unit matrix and $\epsilon(j)$, α and $\beta(j)$ are the diagonal matrices of order M with diagonal elements

$$\epsilon(j)_{mm} = \varepsilon_{jm} \qquad \alpha_{mm} = \frac{1}{\sin q_m} \qquad \beta(j)_{mm} = \frac{\exp(2\mathrm{i}q_m j)}{\sin q_m}. \tag{2.13}$$

The dependence of the transfer matrix on the magnetic flux (which enters through the momenta q_m) will not be indicated but should be kept in mind.

In our expression for the AC MC, a key role is played by transfer matrices $T^{(j)}$ which transform the wavefunction from one side of the system (say the left) up to a column j. Explicitly,

$$T^{(j)} = \prod_{k=1}^{j} \tau^{(k)}. \tag{2.14}$$

Note that the transfer matrix through the entire sample is $T = T^{(j=L)}$. For the computation of the DC conductance, knowledge of T is enough, using the remarkable Pichard [10] formula

$$G_{DC} = \frac{e^2}{h} 2 \mathrm{Tr}[(T^\dagger T) + (T^\dagger T)^{-1} + 1]^{-1}. \tag{2.15}$$

Using the formalism developed earlier [4], we may now derive an expression for the AC MC in the tight-binding model, using the transfer matrices in the plane-wave representation

introduced above. Let us first recall that within the standard linear response theory, in this model, the AC MC $G(\omega)$ is given as [11]

$$G(\omega)=\frac{h}{2L^2}\int \mathrm{d}E\frac{f(E)-f(E+\hbar\omega)}{\hbar\omega}\sum_{ab}\sum_{j=1}^{L}|\langle a|J^{(j)}|b\rangle|^2\delta(E-E_a)\delta(E-E_b+\hbar\omega) \tag{2.16}$$

where $\langle a|$, $|b\rangle$, and E_a, E_b are eigenstates and eigenenergies, respectively, of the entire system and $f(E)$ is the Fermi distribution function at energy E. The current operator is given by

$$J^{(j)}=\frac{eV}{\mathrm{i}\hbar}\sum_{m}(|j+1m\rangle\langle jm|\exp(-\mathrm{i}m\phi)+|jm\rangle\langle j+1m|\exp(\mathrm{i}m\phi)). \tag{2.17}$$

Following the same steps as in [4], we get

$$G(\omega)=\frac{e^2}{h}\int \mathrm{d}E\frac{f(E)-f(E+\hbar\omega)}{\hbar\omega}g(E;\omega) \tag{2.18}$$

where $g(E;\omega)$ has the trace form

$$g(E;\omega)=\tfrac{1}{2}\,\mathrm{Tr}\{\Theta(E)\Omega(E,E+\hbar\omega)\Theta(E+\hbar\omega)\Omega^{\dagger}(E,E+\hbar\omega)\}. \tag{2.19}$$

The $2M\times 2M$ matrix $\Theta(E)$ is defined as

$$\Theta(E)=2[T(E)^{\dagger}T(E)+1]^{-1} \tag{2.20}$$

where we recall that $T(E)$ is the transfer matrix (in the plane-wave representation) through the entire system. Moreover, the dynamic matrix Ω is defined as

$$\Omega(E,E+\hbar\omega)=\sum_{j=1}^{L}T^{(j)}(E)^{\dagger}H^{(j)}(E,E+\hbar\omega)T^{(j)}(E+\hbar\omega) \tag{2.21}$$

in which the $2M\times 2M$ kinematic matrix $H^{(j)}$ is conveniently expressed in terms of its four $N\times N$ blocks:

$$H^{(j)}(E,E+\hbar\omega)=\begin{bmatrix} H_{11}^{(j)} & -H_{21}^{(j)*} \\ H_{21}^{(j)} & -H_{11}^{(j)*}\end{bmatrix} \tag{2.22}$$

which turn out to be diagonal with elements

$$[H_{11}^{(j)}]_{mm}=\frac{\sin[(q_m+q'_m)/2]}{\sqrt{\sin q_m \sin q'_m}L}\exp[-\mathrm{i}(j+\tfrac{1}{2})(q_m-q'_m)] \tag{2.23a}$$

$$[H_{21}^{(j)}]_{mm}=\frac{\sin[(q_m-q'_m)/2]}{\sqrt{\sin q_m \sin q'_m}L}\exp[\mathrm{i}(j+\tfrac{1}{2})(q_m+q'_m)]. \tag{2.23b}$$

The dependence on E and $E+\hbar\omega$ enters through the momenta

$$q_m=k_m+m\phi \qquad q'_m=k'_m+m\phi \tag{2.24}$$

where, as we recall, the momenta k_m and k'_m satisfy (see equation (2.6)) $E_m(k_m)=E$ and $E_m(k'_m)=E+\hbar\omega$, respectively.

Equations (2.18)–(2.24) with the transfer matrices given in equations (2.12)–(2.14) are the central result of the present section. They are completely equivalent to the Kubo formula (2.16) but are much more useful from a practical point of view. It can easily be verified that, as the frequency tends to zero, the Pichard formula (2.15) for the DC conductance is recovered. Naturally, the structure of these equations is more involved

than that of equation (2.15) but the basic ingredients are simply the transfer matrices $T^{(j)}$ (equation (2.14)) at energies E and $E+\hbar\omega$.

As a simple example for the use of the above formalism let us consider a clean system, which in this model is implemented by the absence of site potentials ($\varepsilon_{jm}=0$). This implies that all the transfer matrices are identical with the unit matrix. The AC MC then becomes a sum of independent contributions from the 1d subbands, namely

$$g(E;\omega)=\sum_{m=1}^{M} g_m(E;\omega) \tag{2.25}$$

where

$$g_m(E;\omega)=\frac{1}{\sin q_m \sin q'_m}\sum_{s=\pm 1}\frac{\sin^2[(q'_m+sq_m)/2]\sin^2[(q'_m-sq_m)L/2]}{\sin^2[(q'_m-sq_m)/2]L^2}. \tag{2.26}$$

Equation (2.26) shows that, unlike the DC case, the AC conductance of a clean system depends on the system's length L in a non-trivial way and decays as L^{-2} for large L. Note also that, if one uses the Ohmic relation between conductance and conductivity, the last quantity will decay as L^{-1} for large L, contrary to our intuition. The reason is that the use of the Ohmic relation as it stands is not justified, since it relates a bulk property (conductivity) to a mesoscopic property (conductance) without taking into account the effect of the leads. Note that the basic quantity is measurable, which, in the present context, is the conductance and not the conductivity. This point has already been discussed in earlier work [4, 8].

Before presenting our numerical results it is instructive to extend the above formalism so as to include models with continuous geometry (in contrast with lattice models), where the role of edge states is crucial [12, 13]. We carry it here briefly, since the structure of the equations is almost identical with those presented above. As a typical continuous system we may consider a strip stretched along the x direction, in such a way that electrons are confined in the transverse (y) direction and are subject to a strong magnetic field in the z direction. In the Landau gauge, the clean system possesses translation invariance along the x direction, and the basic solutions are edge states of (quasi-)momentum q:

$$e_n(x,y)=\exp(iqx)f_n(y;q) \tag{2.27}$$

with energy $\varepsilon_n(q)$ in which n is the band index. The number of dispersion curves which intersect the Fermi energy E determines the number N of propagating modes [14] with momenta $\pm q_n$ defined as solutions of the implicit equation $\varepsilon_n(\pm q_n)=E$. For each N there are N edge states going to the right and the same number of edge states going to the left. The AC electric field is directed along the x direction and the leads are separated by a distance L. If the disorder is implemented by short-range potentials located at points (x_i, y_i) such that $0=x_0<x_1<x_2<\ldots<x_K<x_{K+1}=L$, then between two points x_j and x_{j+1} the wavefunction in this slice is a sum of edge states:

$$\Psi^{(j)}(x,j)=\sum_{m=1}^{N}[a_m^{(j)}\exp(iq_mx)f_m(y;q_m)+b_m^{(j)}\exp(-iq_mx)f_m(y;-q_m)]. \tag{2.28}$$

From now on, the definition of transfer matrices is identical with that of equations (2.12)–(2.14) and the evaluation of the AC MC follows (2.18)–(2.24). The only difference occurs in the structure of the kinematic matrix H (whose detailed expression will not be given here). Its four $N\times N$ block matrices are not diagonal (see equations (2.22)–(2.23)) and, as a result, the decomposition of the AC MC into a sum of contributions from one-dimensional subbands (equations (2.25) and (2.26)) is not exact. The presence of both a magnetic field and an AC electric field lead, in the continuous geometry, to coupling between different

modes even without scattering by impurities. The fact that this coupling is absent in the tight-binding model seems to us an artefact resulting from the Peierls [15] substitution implementing the magnetic field in tight-binding models. Yet, this difference is meaningful only for clean systems, since, for disordered systems, scattering by impurities will couple all modes anyhow.

3. Results and discussion

Before introducing disorder, let us start by looking at the AC MC of a clean system and compare the results in the continuous and in the tight-binding models. As a continuous geometry model for which analytical solutions are available we might consider a strip stretched along the x direction in which the transverse confinement is implemented by a harmonic potential $U(y) = \frac{1}{2}m^*\omega_0^2 y^2$. The edge states $e_n(x, y)$ and the dispersion curves $\varepsilon_n(q)$ are known in closed form. It is useful to define the parameters

$$\alpha = \frac{m^*\omega_0}{\hbar} \qquad b = \frac{eB}{\hbar c} = \ell_B^{-2} \qquad \beta^2 = b^2 + \alpha^2. \tag{3.1}$$

Then the functions $f_n(y; q)$ are Gaussians multiplied by Hermite polynomials (shifted by $y_0 = q\ell_B^2$), and for the energies one readily finds the dispersion law

$$\varepsilon_n(q) = (2n + 1)\beta + \frac{\alpha^2 q^2}{\beta^2}. \tag{3.2}$$

The AC electric field is applied along the x direction and the leads are separated a distance L apart.

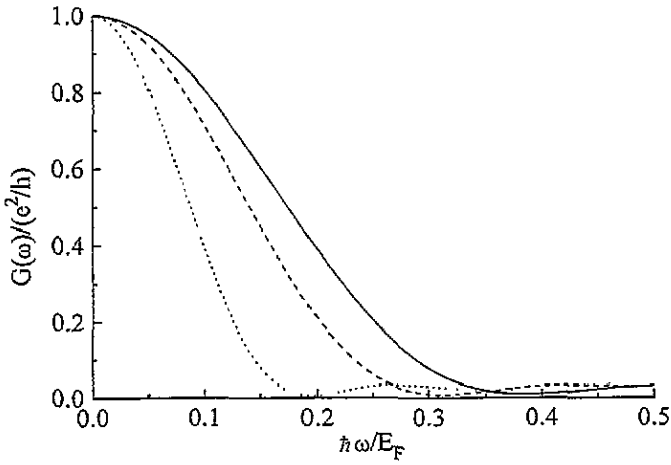

Figure 1. The AC MC $G(\omega)$ (in units of e^2/h) for a clean system in the continuous geometry with parabolic confinement, as a function of the parameter $\hbar\omega/E_F$, where the other parameters are fixed by $L\alpha^{1/2} = 5\pi$ and $k_F\alpha^{-1/2} = \pi/2$: ——, zero magnetic field; – – –, $b = 0.6\alpha$; - - - -, $b = 1.2\alpha$.

In figure 1 we display the AC MC at zero temperature (in units of e^2/h) of a strip whose electrons are confined by such a parabolic potential. The conductance is plotted as a function of the frequency in the range $0 \leqslant \hbar\omega \leqslant 0.5E_f$. The length of a system and the Fermi momentum are fixed by $L\alpha^{1/2} = 5\pi$ and $k_F\alpha^{-1/2} = \pi/2$. The AC MC is calculated at three values of the magnetic field, namely $b = 0$, $b = 0.6\alpha$ and $b = 1.2\alpha$.

For these values of the parameters we have only one propagating channel ($N = 1$). For low frequencies we see that the magnetic field attenuates the conductance, which is to be expected in clean systems. As the frequency becomes higher, the conductance oscillates as a function of the frequency. Later we shall see that the periods of oscillations for different magnetic fields are related to Fermi velocities in the various subbands.

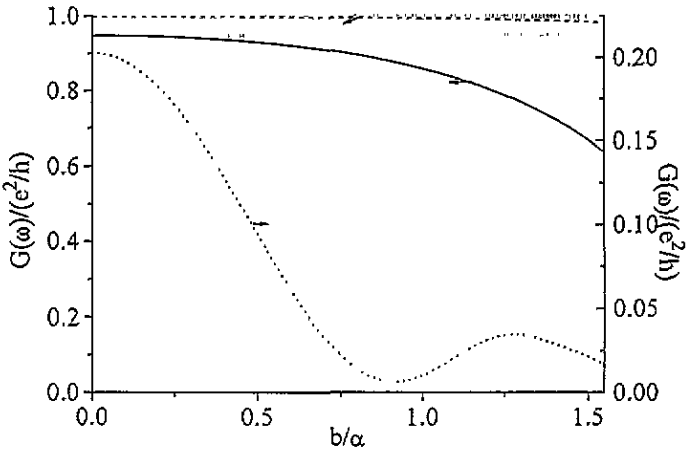

Figure 2. The AC MC for the same system as in figure 1, as a function of b/α: – – –, $\hbar\omega \leqslant 0.01E_F$ (left-hand ordinate); ——, $\hbar\omega \leqslant 0.05E_F$ (left-hand ordinate); ·····, $\hbar\omega \leqslant 0.25E_F$ (right-hand ordinate).

The conductance as a function of the magnetic field in the range $0 \leqslant b \leqslant \pi\alpha/2$ is plotted in figure 2, for which the other parameters are as in figure 1, again leaving a single propagating edge state ($N = 1$). For $\hbar\omega \leqslant 0.01E_F$ the conductance is very close to the DC quantized value. For higher frequencies the slope increases until the conductance starts to oscillate as a function of the magnetic field.

In order to obtain a better understanding of this behaviour let us inspect the AC MC (again at zero temperature) at low frequencies ($\hbar\omega \ll E_F$) where we can get an analytical approximation for it. In the low-frequency limit the main contribution to the AC MC will come from the diagonal parts of the matrices H, with the result

$$G(\omega, B) \approx \frac{e^2}{h} \sum_{n=1}^{N} \frac{\sin^2(\omega L/2v_n)}{(\omega L/2v_n)^2} \exp\left(-\frac{b^2\omega}{4\beta^3 v_n}\right) L_n\left[\frac{b^2\omega}{2\beta^3 v_n}\right] \tag{3.3}$$

where

$$v_n = \frac{\alpha^2 \hbar q_n}{\beta^2 m^*} \tag{3.4}$$

is the Fermi velocity in channel n and $L_n(x)$ is the Laguerre polynomial. Within this approximation, the oscillations can be traced out easily since the conductance is expressed as an incoherent sum of contributions from edge states of different energies. For example, the oscillations can be seen by changing the frequency (the difference between the energy of the edge states). Alternatively, they can be achieved by varying the magnetic field which changes the velocity of the electrons in the various channels. In the zero-frequency limit, one obtains $G = N(e^2/h)$ as expected.

Now let us turn to calculations pertaining to the AC MC in tight-binding models, again considering clean systems first. In figure 3 we plot the AC MC at zero temperature, as a

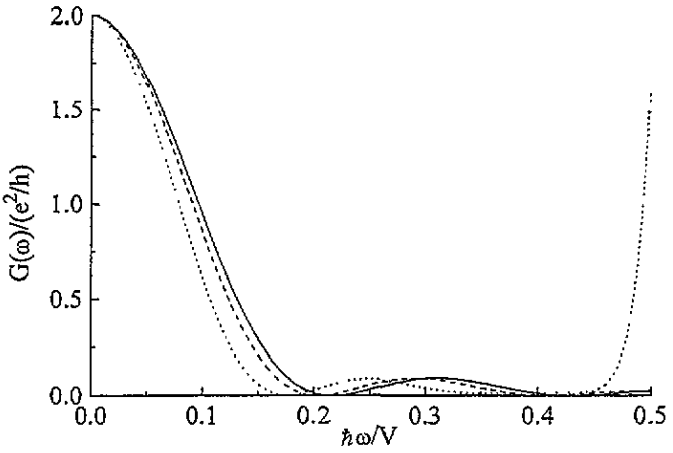

Figure 3. The AC MC in the tight-binding model for an ordered system of length $L = 50$ and width $M = 2$, as a function of $\hbar\omega/V$, where the Fermi energy is taken to be at the band centre: ——, zero magnetic flux per plaquette ($\phi = 0$); – – –, $\phi = \pi/3$; · · · · · ·, $\phi = \pi/2$.

function of the frequency in the range $0 \leqslant \hbar\omega \leqslant 0.5V$, for an ordered system ($\varepsilon_{jm} = 0$) of length $L = 50$ and width $M = 2$. The Fermi energy is taken at the centre of the band so that the number of channels is two. The AC MC is calculated for three values of the magnetic field (expressed here in terms of magnetic flux per plaquette), namely $\phi = 0$, $\phi = \pi/3$ and $\phi = \pi/2$. Very similar to our results for the continuous geometry we can see that near zero frequency the magnetic field attenuates the conductance, but at higher frequencies the conductance oscillates as a function of the frequency.

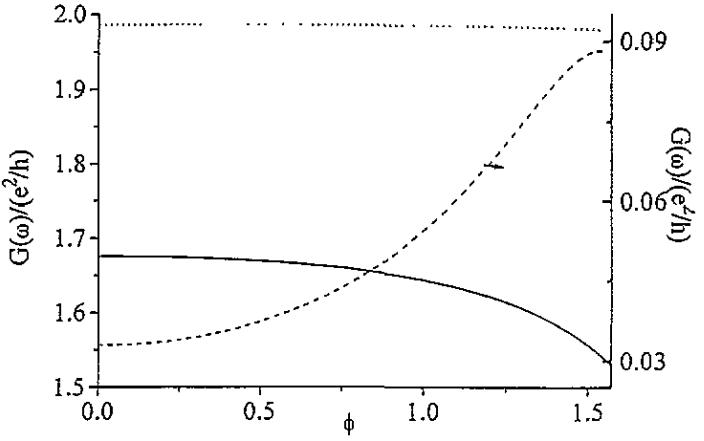

Figure 4. The AC MC as a function of the magnetic field for the same system as in figure 3: – – –, $\hbar\omega = 0.01V$, (left-hand ordinate); ——, $\hbar\omega = 0.05V$ (left-hand ordinate); · · · · · ·, $\hbar\omega = 0.25V$ (right-hand ordinate).

The conductance as a function of the magnetic flux in the range $0 \leqslant \phi \leqslant \pi/2$ is plotted in figure 4, with other parameters as in figure 3, for frequencies $\hbar\omega = 0.01V$, $\hbar\omega = 0.05V$ and $\hbar\omega = 0.25V$. As expected, in the limit of zero frequency we obtain $G = N(e^2/h)$ where N is again the number of occupied subbands (not necessarily equal to M of course). For $\hbar\omega = 0.01V$ the value of the conductance is very close to the DC quantized value and

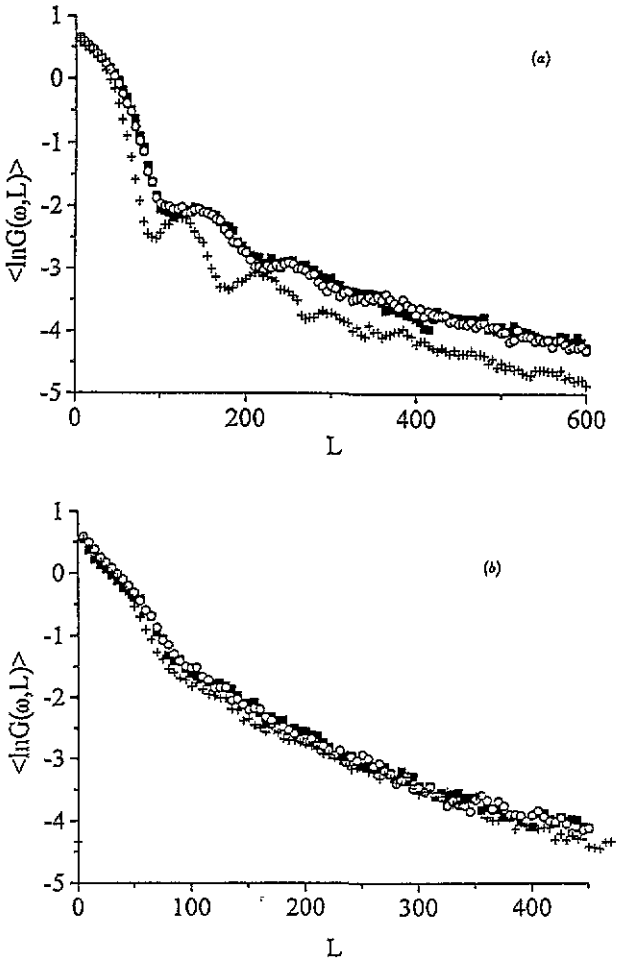

Figure 5. (*a*) The quantity $\langle \ln[G(\omega, L)] \rangle$ calculated in the tight-binding model (for a system of width $M = 2$) as a function of the length of the system L for weakly disordered systems ($w = 0.1V$), for three values of the magnetic field (expressed as flux per plaquette): ■, $\phi = 0$; ○, $\pi/4$; +, $\pi/2$. Other parameters are $E = 0$ (band centre) and $\hbar\omega = 0.1V$. The average is obtained for 500 samples. (*b*) Same as (*a*) for stronger disorder ($w = 1.0V$.)

it is almost a straight line. For higher frequencies the conductance oscillates as a function of the magnetic flux. As in the continuous case, these results can be easily interpreted in terms of equation (2.26).

Finally, let us investigate the AC MC of a disordered system in the tight-binding model. The pertinent questions which we pose are, first, how the AC MC of a weakly disordered quasi-one-dimensional system decays as a function of the system's length L and, second, how the magnetic field affects the conductance compared to its zero-field value. For the DC MC, both questions have definite answers. The conductance decays exponentially with length (the decay rate is determined by the localization length $\xi(E, B)$). Moreover, the theory of weak localization implies that in the absence of spin–orbit coupling the magnetic

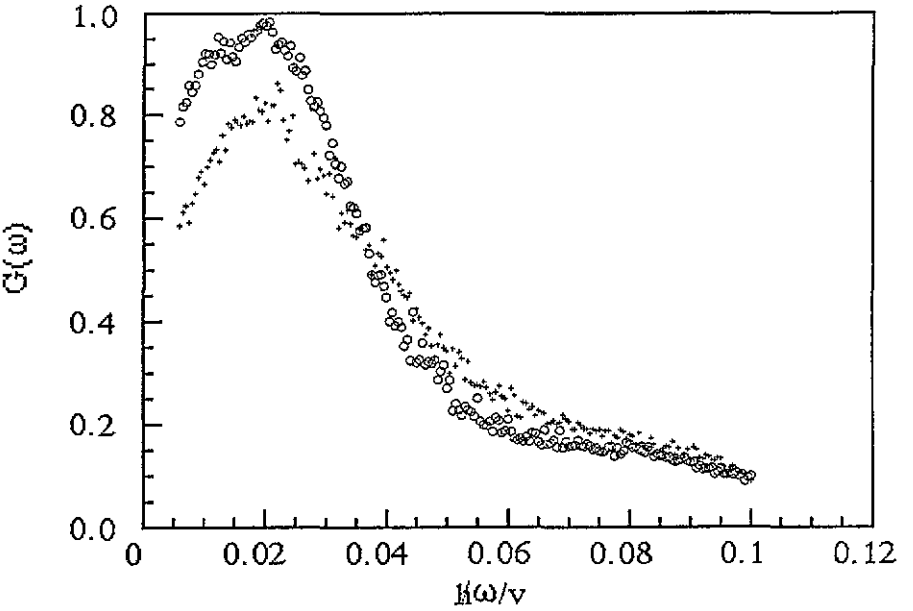

Figure 6. The quantity $\langle G(\omega, L)\rangle$ calculated in the tight-binding model (for a system of width $M = 2$) shown as a function of $\hbar\omega/V$ for a disordered system of length $L = 200$ and for $\phi = 0$ (+) and $\pi/16$ (○). The average is obtained for 500 samples.

field increases the conductance in such a way that [16]

$$\frac{\xi(E, B)}{\xi(E, 0)} = \frac{2M}{M+1} \tag{3.5}$$

where M, we recall, is the width of the system. For the dynamic conductance, the situation is not so clear. First, evidence is mounting which shows that neither $\langle G(\omega)\rangle$ nor $\langle \ln[G(\omega)]\rangle$ is a self-averaging quantity. The question of whether one can define a frequency-dependent localization length has occupied us for some time, and no definite answer has been found. Numerical calculations strongly suggest that $\langle \ln[G(\omega)]\rangle \sim -\ln L$, but there is so far no general proof for this conjecture. As for the second question, we expect that the presence of time-dependent fields destroys the arguments of weak-localization theory since the role of the magnetic field as a time-reversal violating force is not unique.

In figures 5 the first question is addressed. In figure 5(*a*), the disorder is taken to be weak ($w = 0.1V$). The quantity $\langle \ln[G(\omega)]\rangle$ is plotted as a function of L for fixed ω and for $\phi = 0$, $\pi/4$ and $\pi/2$. The magnetic field tends to decrease the conductance and, when the magnetic field is strong ($\phi = \pi/2$), there are regular oscillations, reminiscent of the clean system. For such a strong field, the disorder appears to be just a small perturbation. The situation is different in the medium disordered case ($w = 1.0V$) displayed in figure 5(*b*). In this case the magnetic field has almost no effect at all. It is also evident that $\langle \ln G\rangle$ is not proportional to $-L$, i.e. it is not self-averaging. Rather, the conjecture $\langle \ln[G(\omega)]\rangle \sim -\ln L$ is more appropriate here.

Finally, in figure 6 we address the second question and plot $\langle G(\omega)\rangle$ as a function of $\hbar\omega/V$ for a disordered system of length $L = 200$ and for two values of the magnetic field, namely $\phi = 0$ and $\pi/16$. It is evident that at small frequencies the positive MC prevails, whereas it is washed out completely at higher frequencies.

4. Summary

We have developed a formalism for the evaluation of the AC MC in quasi-one-dimensional mesoscopic systems in terms of the system's transfer matrices. Instead of one transfer matrix which transforms the wavefunction from one side of the system to the other side we need all partial matrices which transform the wavefunction from one point of the system to another point. The central equations which can be used for actual calculations (and which seem to be relatively simple) are (2.18)–(2.24). They are developed for tight-binding models, but similar expressions hold also in continuous geometry models. The present formalism can be considered as an attempt to translate the Kubo formula for the response functions at finite frequency into the Büttiker–Landauer formulation. It reduces to the Pichard formula when the frequency vanishes. Its limitations have already been stressed in the introduction, but nevertheless its simplicity and scope of applications enable one to use it in numerous physically interesting cases.

Employing the above formalism, the AC MC is then calculated for several representative cases. It is found that, even for clean systems, the structure of the AC MC is not trivial, although it can be written as a sum of independent contributions from the one-dimensional subbands. This structure leads to an oscillatory behaviour of the form $(\sin^2 x)/x^2$ where $x = \omega L/v$ in which v is the Fermi velocity. Moreover, in contrast with the DC MC, the AC MC of a clean system at finite frequency depends on the length of the system and behaves as L^{-2} at large L.

For disordered systems, two points are to be remembered. First, at a finite frequency, we cannot spot a self-averaging quantity, and we are tempted to conjecture that $(\ln G)$ is proportional to $-\ln L$, as is evidently suggested in figure 5(b). Second, one cannot speak of a positive (or negative) AC MC since it is not valid at higher frequencies. This is evident from a glance at figure 6.

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